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ARITHMETICAL AND ALGEBRAICAL APPROACH TO THE SECOND DEGREE EQUATIONS[1] (AGE: 15 − 19 YEARS)

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Abstract. The principal aim of this work is trying to observe/describe in which way the slovak students approach the study of algebra and in particular the second degree equations, underlining some differences and similitude to the italian students` approach to the same subject.
This study is inserted in a more big research started two years ago in Italy with a the thesis of Benedetto Di Paola on Didactic of Mathematics in Palermo (Supervisors: Prof. T. Marino, Prof. F. Spagnolo) (http://math.unipa.it/~grim/). In his thesis, according to the idea that the algebraic thought developed in the History of the Mathematics only after the construction of the arithmetic thought (as for the Western culture), Benedetto Di Paola analyzed a particular didactic experience on the students of secondary school, according to the theory of situations of Guy Brousseau (Brousseau, 1997, 1998) with a semiotic revision in this connection with interpretations of phenomena of learning/teaching (Spagnolo, 1998) as for the methodology of the study: analysis a priori and final results (quantitative analysis) of the presented subject.
In this paper we want to analyze some of the results observed realizing the same didactical experience with the Slovak students, during our permanence in Slovakia, and to compare these with some of the most interesting results of the research in Palermo.
This work follows two essential lines: first of all we will briefly report some historical notes about the History of Algebra and therefore the development of the algebraic thought underling the importance of the historical point of view in an experimental research like this, secondly, in the second part of the paper, the most important one, we will analyse the didactical experience (problematic situation) on the italian[2] and Slovak[3] students, report-

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ing the analysis a-priori of the test used for the didactical experience and presenting the analysis a posteriori of the final results (quantitative analysis)..

Résumé. L´objectif principle de ce travail est essayer décrire comment les étudiants slovaques s´approchent de l´algebra, particulièrement d´équations de second degrés, en soulignant des differences et des analogies aux étudiants italiens.
Cet étude est une partie d´une recherche grande qui s´a commence il y a 2 ans en Italie par la these de Benedetto Di Paola à la Didactique de Mathématiques à Palermo (sous la conduite de Prof. T. Marino, Prof. F. Spagnolo) (http://math.unipa.it/~grim/).
Dans sa these, en correspondant à l´idée que l´approche algébraique s´était développé dans la histoire de mathématiques seulement après la construction de l´approche arithmétique (quant à la culture occidentale), Benedetto Di Paola a analysé une expérience didactique particulière chez éleves de l´école sécondaire, conformément à la théorie de situations didactiques de Guy Brousseau (Brousseau, 1997, 1998) avec une revision semiotique par rapport à l´interprétation du phénomène de l´apprentissage /l´einseigne-ment (Spagnolo, 1998) quant à la methodologie de l´étude: analyse a priori et les résultats finals (l´analyse quantitative) du subject presenté.
Dans cet article on veut analyzer quelques résultats obtenus de l´experience didactique chez éleves Slovaques pendant notre presence en Slovaquie et comparer ceux-ci avec des résultats les plus importants de la recherche à Palermo.
Ce travail suivit deux lignes essentiales: dans la première on va rapporter brièvement des notes historiques de l´Histoire de l´Algebra et puis le développement de l´approche algébraique en soulignant l´importance des issues historiques dans une recherche expérimentale come celle-ci, après dans la deuxième – la plus importante - partie de cet article, on va analyser l´expérience didactique (la situation de problème) chez éleves italiens et slovaques, en rapportant l´analyse a-priori du test appliqué pour l´expériance didactique et en présentant l´analyse a posteriori des resultats finals (l´analyse quantitative).

Zusammenfassung. Das Hauptziel dieser Arbeit ist beschreiben wie slowakische Studenten an das Studium der Algebra herangehen zu versuchen. Speziell widmen wir uns den Gleichungen zweiten Grades, wobei wir auf manche Verschiedenheiten und Ähnlichkeiten mit dem Herangehen italienischer Studenten zum Thema konzentrieren.
Diese Studie ist Bestandteil einer größeren Forschung, die vor zwei Jahren in Italien begann und ist ein Teil der Doktorarbeit von Benedetto Di Paola, Doktorand am Lehrstuhl der Didaktik der Mathematik in Palermo (unter Aufsicht von: Prof. T. Marino, Prof. F. Spagnolo) (http://math.unipa.it/~grim/).
In seiner Doktorarbeit, im Einklang mit der Idee, dass das algebraische Denken in der Historie der Mathematik erst nach dem Ausbau des arithmetischen Denkens entwickelt wurde (was die westlichen Kulturen anbelangt), analysierte Benedetto Di Paola eine konkrete didaktische Erfahrung der Studenten der Mittelschule, im Einklang mit der Theorie der didaktischen Situationen von Guy Brousseau mit einer semiotischen Kontrolle im Zusammenhang mit der Interpretation der Phänomens Lernen/Unterricht (Spagnolo, 1998) und Analyse a priori, Kvantitativanalyse – Endresultate des präsentierten Thema was die Methodologie des Studiums anbelangt.

[3] In collaboration with the University of Bratislava (Supervisor: I. Trenčanský, Tutor: H. Bereková, Teacher: J. Žabka) and the teachers of the school where we worked/studied during our mobility of "*Socrates Comenius* European Teacher",

In diesem Artikel wollen wir manche bei slowakischen Studenten während unseres Slowakeiaufenthalts mit gleicher didaktischer Erfahrung erlangten Resultate analysieren und die Analysenresultate mit den interessantesten Resultaten der Palermoforschung vergleichen. Die Arbeit beinhaltet zwei Grundlinien: In Erster Linie werden Notizen aus der Historie der Algebra und die Entwicklung des algebraischen Denkens, vor allem die Wichtigkeit eines historischen Ausgangspunkts in Forschungen wie dieser, dargestellt.
In Zweiter Linie – dem eigentlichen Grund dieses Artikels, werden wir die didaktische Erfahrung (eine Problemsituation) bei italienischen und slowakischen Studenten analysieren. Dazu wurde die Analyse a priori des benützten Tests der didaktischen Erfahrung vorgelegt und eine Analyse a posteriori der Endresultate ausgeführt.

Riassunto. Scopo principale del presente lavoro è quello di tentare di osservare/descrivere in che modo gli studenti slovacchi affrontano lo studio dell'algebra ed in particolare le equazioni di secondo grado, sottolineando differenze e somiglianze con l'approccio seguito dagli studenti italiani sullo stesso tema.
Questo studio è inserito in una più grande ricerca iniziata due anni fà con la tesi in Didattica della Matematica di Benedetto Di Paola, Palermo (Supervisors: Prof. T. Marino, Prof. F. Spagnolo) (http://math.unipa.it/~grim/). Nella sua tesi, avendo, sposato l'idea che il pensiero algebrico si è sviluppato nei secoli soltanto dopo che si è enucleato il pensiero aritmetico (almeno per la cultura occidentale), Benedetto Di Paola ha analizzato una particolare esperienza didattica svolta con gli studenti della scuola secondaria superiore in accordo con la teoria delle situazioni di Guy Brousseau (Brousseau, 1997, 1998) con una revisione semiotica in relazione con il fenomeno di aprrendimento/insegnamento (Spagnolo, 1998), per quanto attiene alla metodologia di studio seguita: analisi a priori e risultati finali (analisi quantitativa) dell'argoment trattato.
Nel presente articolo vogliamo analizzare brevemente alcuni dei risultati ottenuti realizzando la stessa esperienza didattica con gli studenti slovacchi, durante la nostra permanenza a Bratislava, e comparare questi con alcuni dei risultati più interessanti evidenziati nella ricerca condotta a Palermo.
Il lavoro seguirà principalmente due linee fondamentali: nella prima parte riporteremo brevemente alcune note storiche sulla storia dell'Algebra e quindi sullo sviluppo del pensiero algebrico sottolineando l'importanza dell'approccio storico in un lavoro sperimentale quale questo nostro, successivamente, nella seconda parte dell'articolo, la parte più importante, analizzeremo l'esperienza didattica (situazione-problema) condotta con studenti palermitani e slovacchi, riportando l'analisi a priori del test usato per l'esperienza didattica e presentando l'analisi a posteriori dei risultati finali (analisi quantitativa).

Abstrakt. Hlavným cieľom tejto práce je pokúsiť sa popísať akým spôsobom pristupujú slovenskí študenti k štúdiu algebry, špeciálne k rovniciam druhého stupňa, zdôrazňujúc niektoré rozdiely a podobnosti s prístupom talianskych študentov k danej téme.
Táto štúdia je súčasťou rozsiahleho výskumu, ktorý sa začal pred dvomi rokmi v Taliansku a je súčasťou dizertačnej práce Benedetta Di Paolu, doktoranda katedry Didaktiky matematiky v Palerme (Školitelia: Prof. T. Marino, Prof. F. Spagnolo) (http://math.unipa.it/ /~grim/). V jeho dizertačnej práci, v súlade s myšlienkou, že algebraické myslenie sa v histó-rii matematiky rozvíjalo len po vybudovaní aritmetického myslenia (pokiaľ ide o západné kultúry), Benedetto di Paola analyzoval konkrétnu didaktickú skúsenosť u študentov strednej školy, v súlade s teóriou didaktických situácií Guya Brousseaua (Brousseau, 1997, 1998) so semiotickou kontrolou v súvislosti s interpretáciou fenoménu učenie/vyučovanie (Spagnolo, 1998) pokiaľ ide o metodológiu štúdia: analýza a priori a konečné výsledky (kvantitatívna analýza) prezentovaného námetu.

V tomto článku chceme analyzovať niektoré výsledky získané rovnakou didaktickou skúsenosťou u slovenských študentov, počas našej prítomnosti na Slovensku, a porovnať tieto výsledky s niektorými najzaujímavejšími výsledkami výskumu v Palerme.
Táto práca sleduje dve základné línie: v prvej v krátkosti uvedieme poznámky z Histórie Algebry a vývoj algebraického myslenia vyzdvihujúc dôležitosť historického východiska vo výskume ako je tento, ďalej v druhej – najpodstatnejšej – časti článku, budeme analyzovať didaktickú skúsenosť (problémovú situáciu) u talianskych a slovenských študentov, predložiac analýzu a-priori testu použitého pre didaktickú skúsenosť a prezentujúc analýzu a posteriori konečných výsledkov (kvantitatívna analýza).

Key words: experimental analysis, analysis a priori, analysis a posteriori, second degree equations, arithmetical approach, algebraic approach

1 Some historical notes

To solve algebraic questions and so second degree equation could seem simply because the only things that a student has to know are: literal calculation, Algebra and analytic Geometry of first degree equations and calculation with radicals; but it isn't true.

Some experimental studies (Harper, 1987; Sfard, 1992) seem to confirm the piagetian thesis of convergence between historical development and individual development (Garcia and Piaget, 1989): the difficulties of a student could be very near to the problems that generations of mathematicians have experimented. If we analyse the history of Algebra (in particular the different procedures of resolution of the equations) underling the History of the construction of the algebraic language in correlation with the History of Arithmetic we could notice that the development of this theory was very slow and very hard (Malisani, 1996).

If we consider this point of view, the History of Algebra could shows us "why", for a student, isn't simply to learn and to use Algebra and so in particular the concept of equation.

The preliminary study of the epistemological and historical-epistemological representations, result fundamental then to be compared with the experimental contingency and the more deepened this kind of analysis is, the more possible to deduce the phenomenon of teaching/learning and to reproduce it in other analogous conditions will be (Spagnolo, 1998).

In this vision, wanting to delineate, in a first approximation, a historical picture of the development of Algebra and therefore of the symbolic system used for expressing the proper concepts of this discipline, with the purpose to underline the most important tappets of the evolution of the algebraic thought, we could refer to the studious G.H. Nesselmann.

G. H. Nesselmann (Cfr. Arzarello, 1994; Nesselmann, 1843) individualizes three different stadiums in the history of Algebra:

– "*Rhetorical* " (Before Diofanto of Alessandria, 250 d.C.)
  An oral Algebra, all to words, without symbols. A typical example is the Babylonian mathematics.

– "*Syncopated* " (from Diofanto to the end of XVI Century)
  Algebra has some more algebraic notations (unknown quantities, equality, powers, etc.), as in Diophantus.

– "*Symbolic"* (Viète, 1540-1603)
  Algebra is the modern notation, introduced by Viete and Descartes, in which lexicon and syntax are specifically designed, and, most of all, there are rules of symbolic manipulation to rewrite the expressions. Such rules in the earliest versions were simply 'written' and 'argumented' and the idea of proof was 'rigorous' but not 'algorithmically defined'.

These three stadiums sign a development of the Algebraic Thought.

A deep lecture of these three different stadiums, with particular attention to the study of some problems introduced in the different books written in the various historical epochs as those Babylonian, Greek, Egyptians, Indian, Arabs and Europeans, could give us then the possibility to analyze the development of the mathematical knowledge related to the equations of first and second degree.

Making an example, in the study of the epistemological obstacles with regard to the algebraic language and in the analysis of the binomial Arithmetic-Algebra relatively to the equations, we could refer to:

– the problem N.24 of the *Rhind's papyrus* **as an example of a purely arithmetic method of resolution**: the method of solution "rule of false position", of *Ahmes* it isn't anything else than a purely arithmetic method of resolution;
– the problem N.27 of the *Arithmetica* of *Diofanto* in witch the solution of the problem of second degree rotates around the unknown auxiliary z. The appearance of this unknown assistant, the "aritme", constitutes, together with the symbols a real conceptual change. Is it possible in fact to affirm that the history of **Algebra starts just with Diofanto**.

The analysis of these two historical problems, seemed very interesting for our research: the idea of convergence between the historical development and the individual one seems, in fact, indeed central in this case. The two different strategies of solutions underlined by the Ahmes and Diofanto, arithmetical strategy and algebraic strategy respectively, seemed very near, very similar in many aspects, to those used by the students during the experimentation of the test in the two research in Palermo and Bratislava.

2 EXPERIMENTAL ANALYSIS (ANALYSIS A PRIORI AND ANALYSIS A POSTERIORI)

As already we observed isn't easy to affirm if a pupil possesses, certainty, the algebraic thought or not.

It is interesting, then, according to us, to test, to investigate, on the behaviour of the students to approach an algebraic problem, underlining step by step the various reasoning to search the solution of a question.

Which behaviour will have a pupil, in front of a mathematical question, introduced in the natural language? Will he remain to the arithmetic level, using only the proposed language and therefore expressing the solution "to words"? Will he do, instead, the breakthrough that would bring him to put in formula the problem and so to use, therefore, the algebraic language, succeeding to check the syntax of the proposition?

Working, instead, with a typical algebraic question, will he be more uninhibited in the resolution? Which types of error will he commit?

As already we said, the research that we conducted, had at the base of it the use of a test, the same test that Benedetto Di Paola prepared for his thesis in Palermo (appendix A).

It presents seven questions expressed in different languages: someone in arithmetical/natural language, other in pure algebraic language.

The first type of problem presents therefore a succession of information typical of orally speaking and therefore apparently simple.

The second type is characterized, instead, by a symbolic language. These kinds of problems, as for example the resolution of an algebraic expression, are the more classical questions of Algebra.

The principal goal, presenting this kind of test, was to have the possibility to try and observe the different strategies to approach the Algebra and in particular the second degree equations, underlining the capacity to solve a problem in relationship to the type of writing in which it was given.

During our permanence in Bratislava on March 2004 we introduced this test to many students of inclusive age between 15 and 19 years. We had been involved in total 60 students of Bratislava.

The classes in which we introduced the text were: one IV class (*4.A class* half class 7 students), one VIII class (*Riders class*, half class 8 students) one *Zillions class* (24 students) and one *Guardians class* (21 students) of the school " Sukromne` gymnasium".

We ask to the students to solve (in maximum 35 minutes) the exercises, underlining the various passages to find the solution.

We could assume that, when a student read and understand an exercise in the test, maybe he will solve it following a quite precise strategy (but this assume isn't strictly necessary for this kind of research). His procedural techniques could show us if he has the arithmetical thought and if this one took root on him in such a way to bind him in the algebraic resolution or if he is capable to think in a algebraic way and therefore he is able to translate the test in algebraic way and solve the questions with algebraic methods.

2.1 ANALYSIS A PRIORI

The Analysis a priori is a quite precise tool of investigation and it has a double valence:

- An important valence on the behaviour of the involved students,
- A valence on the discipline in connection to the contents.

The Analysis a priori is necessary in this kind of work because we need not only to analyze, more accurately, the behaviour of the students but also to render more objective the evaluation of the test proposed.

The main purpose of the Analysis a priori is to sort and to restore all the possible strategies applied from the students in the test.

Therefore, in reference to our work, we report the analysis a priori made by Benedetto Di Paola in his research in which it is possible to read some of all the possible/plausible methods of resolution for every question given in the test, correct and not correct; divided in **algebraic strategies (Al)** and **arithmetical strategies (Ar)**.

Compiled the Analysis a priori and somministrated the test, we confronted the hypothesized results with the results gave by the students with the experimental analysis and, at the end, we pulled the final conclusions.

To report the data related to the experimentation, for each of the hypothesized strategies we followed the same technique adopted in the previous research of Benedetto Di Paola. So we built in an Excel's page two columns: the column A that showed if a student, following a determined method of resolution, developed the question and the column B that underlines if he didn't solve the exercise in a complete form and therefore he didn't reached the requested result or if he committed some mistake of calculation.

We report all the hypothesized strategies that we considered (correct strategies):

Ar11A	The students detaining himself on the arithmetic operation of the multiplication, performs a series of calculations (repeated multiplication): 27*1; 28*2; 29*3; 30*4; 31*5;……; 36*10; 37*11; 38*12=456 He departs from the product 27 x 1 and he follows this procedure until he finds, as result of the multiplication, the number 456. The first factor represents the age of the father, the second, instead, the age of the child. The actual age of the father is 38 and the age of the child 12. This strategy introduces therefore only a series of elementary operations without symbolism and abstraction.

Ar12A	The student, to solve the exercise, draws a figure composed by a square with side x and a rectangle with dimensions (26;x) and decides to work with the areas. To find the unknown x, he will have to calculate the area of the square A. (area of the square x^2). He knows that the total area of the figure is 456 and so to calculate the area of the square, he has to subtract to 456 the area of the rectangle B (area of the rectangle "base for height" therefore, in this case: 26 * x) He proceeds for attempts calculating: 456- (1*(26)) ; 456- (2*(26)); 456- (3*(26)); …. ; 456- (11*(26)) ; 456- (12*(26))=144=12^2 Since he has a square, the square of 12, he is sure that x is 12. The child is therefore 12 while the father is 12+26=38 years old. We could classify this strategy as arithmetical-geometrical. The student to solve the question, resorts to the Geometry and after he proceeds for attempts.
Ar13A	The third hypothesized strategy is similar to the second because it's also founded on the calculation of the area, but in this case the figure is not divided into a rectangle and a square but considered as an only great rectangle C. The student who decides to solve the exercise following this methodology to find the result, has to calculate the area of this figure (area of the rectangle "base for height" therefore in this case: (x+26)*x) He does it, using the arithmetic thought, proceeding, as in the previous case, for attempts. (1+26)*1 ; (2+26)*2 ; (3+26)*3 ; ….. ; (11+26)*11 ; (12+26)*12=456 When he finds the result proposed in the text, he discovers that the unknown value of the x is 12. The child is therefore 12 while the father is 12+26=38 years old.
Al11A	The student, to solve the proposed question, decides to formalize it in algebrical language. He chooses therefore to put in formula the problem. He writes then the algebraic equation representing the text of the exercise: (x+26)*x=456. The unknown represents the age of the boy. He solves this equation according to a studied method. Discarding the solution x2=-38, just because the age necessarily has to be >0, he finds the age of the boy: he is 12 years old while his father is 26+12=38. This strategy underlines a certain symbolism and an abstraction. The use of the variable could point out the passage from a pure arithmetic thought to a pre-algebraic thought.
Al12A	The student, to solve the proposed question, decides, in this case, to formalize it in the algebraic language. Then he translates in a formula the problem. He builds, therefore, a system that represents the question and he solves it following one of the learned methods. $\begin{cases} x - y = 26 \\ x * y = 456 \end{cases}$ The unknown x represents the age of the father, the y the age of the child. The boy, then, is 12 years old while his father 38. This strategy is the strategy that indeed show the algebraic way to think. Following this methodology he effects a notable abstraction that brings him to the translation with a formula the problem.
Ar21A	The student works for following approximations. 25*1 ; 25*2 ; 25*3 ; … ; 25*8 ; 25*9 ; 25*10=250 When he has the last result: 25*10=250 , he rereads the text and he effects the subtraction 250-10=240. This is the number proposed in the exercise. He realizes, therefore, that the searched number of balls is indeed 10. In fact: (25€*10)-10€=240€.

Ar22A	The student works with a repeated addition. 25+25=50; 25+25+25=75; …+…; 25+25+25+25+25+25+25+25+25+25=250 With the sale of 10€ on the general price, the team was able to buy 10 balls.
Ar23A	The student works with a repeated subtraction. 240-25=215 ; 240-25-25=190 ; …-…. ; 240-25-25-25-25-25-25-25-25-25=15 When he has the last result 15, he reflects on the sale. With the sale of 10€ on the general price, the team is able to buy one more ball, infact 15€ + the sale of 10€ =25€ and it is the cost of a ball. Then the team, with this sale, buy in total 10 balls.
Ar24A	The student solve the exercise working on the concepts of "unitary cost", "general cost" and "discount". Then he will consider the arithmetical formula: (240+10)/25=10. He will divide therefore the total cost plus the sale by the unitary cost of the balls. He will find so the number of balls bought by the team.
Al21A	The student “translates” the problem with a formula writings this equation: 25x-10=240 He found the value of x that means the number of bought balls.
Al22A	The student “translates” the problem with a mathematical formula writings an algebraic proportion, putting in evidence the cost of one ball and the total cost: he write so: 25(unitary cost) : 1(one ball) = T (total cost) : X But how much is T? To calculate T, the student sums 240,00 € plus the sale: 10 € and so 250 €.
Ar31A	The student proceeds by following attempts. In the first time he takes the numbers 1, 2, 3. These are three integer consecutive numbers. He experimentally tries if the square of the first one plus the square of the second one plus the third square gives indeed the demand result 50. He realizes however that $1^2+2^2+3^2=14\neq50$. Then he repeats the same procedure for 2,3,4 . $2^2+3^2+4^2=29\neq50$. The third attempt could be 3,4,5. $3^2+4^2+5^2=50$. The student realizes that he finds the requested result: 50. The consecutive integers number demanded are: 3, 4, 5. The student doesn’t verify if some others number can resolve the equation. He finds these and he decides to stop himself.
Ar32A	The student follows the previous strategy but instead of working with natural positive numbers he decides to work with negative numbers and so he will try with: (-3, -2, -1); and after with (-4, -3, -2) and so on. He will find (-5,-4,-3) after many attempts As the previous strategy, maybe the student will not ask himself is there could be other numbers that resolve the equation.
Ar33A	The student follows the strategy Ar31A or the strategy Ar32A, but he chooses the three consecutive integer number without a logical sequence. We could assume that working in this way he spends much time to find the solution of the problem.
Al31A	The student “translates” with a formula the problem and so he writes: $x^2+(x+1)^2+(x+2)^2=50$. He solves it using one of the learned methods. Following this strategy, he is able to find all the solutions of the equation and so: (3, 4, 5) ed (-5, -4, -3).
Al32A	The student decides, also in this case, to formalize the problem with the algebraic language. He translates, then, the problem with a formula and so he uses and solves a system like this: $\begin{cases} x^2 + y^2 + z^2 = 50 \\ y = x+1 \\ z = x+2 \end{cases}$

Ar41A	The student proceeds by following attempts. Departing from the number 1 he performs the operations described in the text of the problem: sum 1+4, subsequently multiply the gotten result for 80 and controls if the gotten number is indeed the number required by the exercise. If he has 2.360, as result, he stops himself and he shows the unknown x, otherwise he performs the following step considering a number greater than 1. He will work, therefore, equally for 2, for 10, for 15, for 20, for 25; getting however numbers smaller and smaller than 2.360. Is it possible to suppose that he will consider the number 26. For the number 26 we have the following situation: 26+4=30; 30*80=2.400. But 2.400> 2.360. For 25 we have, therefore, the number 2.320; for 26, instead, the number 2.400. Since, between the two considered numbers there are not any other integer numbers and so it isn't certainly true that if to an integer number we sum 4 and then we multiply it by 80 we will have 2.360. Such integer number doesn't exist.
Ar42A	Another strategy that a student could use to solve the exercise is to draw a rectangle with dimensions ((x+4);80) and work with the areas. The student who works in this way, to find the result has to calculate working with the area of the rectangle, the measure of the base of this rectangular. The area of the rectangular is 2.360. He works using the arithmetical thought, proceeding by following attempts. (1+4)*80 ; (2+4)*80 ; (3+4)*80 ; ….. ; (24+4)*80 ; (25+4)*80=2.320; (26+4)*80=2.400 > 2.360. For the number 26 we have the following situation: 26+4=30; 30*80=2.400. But 2.400> 2.360. For 25 we have, therefore, the number 2.320; for 26, instead, the number 2.400. Since, between the two considered numbers there are not any other integer numbers
Al41A	The student "translates" with a formula the problem and so hewrites: (x+4)*80=2.360
Al42A	The student draw a rectangle with dimensions: base (x+4) ed height 80: ((x+4);80 The student, to solve this exercise, decides to write a equation: 2.360= b*h. and so: 2.360=(x+4)*80 he finds the same conclusion of the previous strategy: Al41A
Ar51A	The student reads the algebraic expression and chooses to solve it proceeding by following attempts. The student, solving the exercise in this way, will do many calculations and he will spend much time to find the solution.
Al51A	The student reads the algebraic expression and solves it with learned methods.
Ar61A	The student reads the algebraic expression and chooses to solve it proceeding by following attempts. The student, solving the exercise in this way, will do many calculations and he will spend much time to find the solution.
Al61A	The student reads the algebraic expression and solves it with learned methods.
Al62A	The student doesn't observe in a good way the equation $x^2/4-x/32=0$ and he solves it mechanically without reasoning. So he will calculate the Delta and he will find the two solutions.
Ar71A	The student decides to work proceeding by following attempts and building same particular cases. He will follow this kind of reasoning: << If the teachers are 10, How many students are in the school?….6*10=60. If the teachers are, instead, 20 then the students are …….6*20=120 >>. Working with this kind of examples the student could find the final formula: S(students) = 6 * P(teachers)
Al71A	The student "translates" with a formula the problem and so he writes: S = 6 * P.

2.2 ANALYSIS A POSTERIORI (QUANTITATIVE ANALYSIS) AND CONCLUSIONS

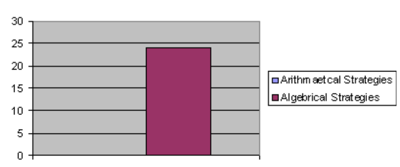
As the Italian student, the Slovak students were very interested in the test and they were hocked for resolving the assignment. As in the previous research made in Palermo for the same didactical experience, we recommended to the Slovak student to write not only the result of the question but also to write a sort of explanation about their solution and their algorithmic to find the solutions. In this way we were able to check the performed strategy and therefore to classify it.

We considered not valid the answers in witch the students didn't write the followed procedure. As would we classify these? How did the student found these solutions, what kind of reasoning did he follow to reach that solution? To facilitate the consultation of the received results about the experimentation, we added to the various tables, built by the software Excel, some bar graph that show in an intuitive way the "prevalence" of the algebraic thought on the arithmetician thought or vice-versa.

To try to obtain the main goal of this research and so to allow a sort of comparison between the Italian and the Slovak students, we compared all the tables prepared for each one of the question of the test. The strategies that we individualized in the analysis a priori are shown valid and exhaustive because the hypothesized behaviours occurred. The only strategies hypothesized that we didn't find in the experimental analysis of the both researches were the Arithmetic-Geometric strategies. None of the students in fact used the Geometry for the resolution of the questions.

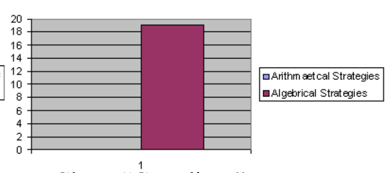
From the analysis of the picked data we could notice a different behaviours of the students according to the logical structure of a problem. We noticed that in an algebraic environment (questions. 5,6,7), almost all students appear more uninhibited and more skilled, succeeding therefore to solve, for example, an algebraic equation in autonomous way and so showing a strong knowledge of the algebraic language.

Question 5



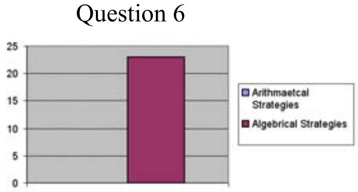
Class: "Zillion"

Question 5



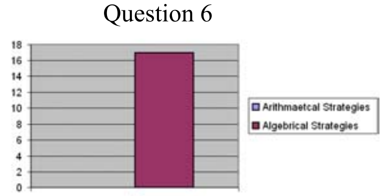
Class: "Guardians"

Question 6



Class: "Zillion"

Question 6



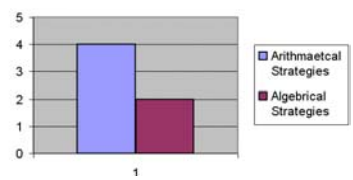
Class: "Guardians"

We don't say the same for all the exercises just because, for example in the first question, many students use arithmetical strategies to solve it, maybe because they see this kind of question not as an algebraic question in witch is possible to translate the natural language into an algebraic language and solve it using a second degree equation.

It seems to confirm us that the algebraic thought is not an aware thought in the student. They didn't reach, in this stadium, the algebraic thought; they stopped at a previous step, a step that we can call pre-algebraic thought.

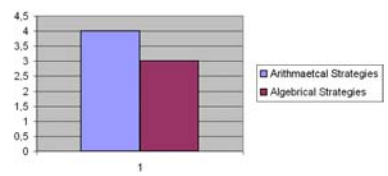
If the had been an aware thought infact, it would be also due out in the resolution of other questions proposed in the test with a not properly algebraic formulation.

Question 1



Class: "Riders"

Question 1



Class: "4.A"

In an arithmetical environment, for example in the first one question, in witch the exercise was proposed in a natural language, the student preferred to use arithmetical strategy instead of algebraic strategies.

We could notice therefore particular situations that show us as to the secondary school, even if the students have a series of algorithms and strategies contemplated to favour the algebraic thought, *they don't reach the formal algebraic thought* but they stop themselves at a previous step, at a pre-algebraic thought.

This kind of result is very interesting for us because reading the result of Benedetto Di Paola about the same research made in Italy *(http://math.unipa.it/~grim/)* we could notice that it is possible to find a sort of correlation between these researches. We have to say, moreover, that, of course, not all the results are the same in these two country, just because, for an example, the two school systems are different and so the approach to this kind of subject that we analyzed with this research is different.

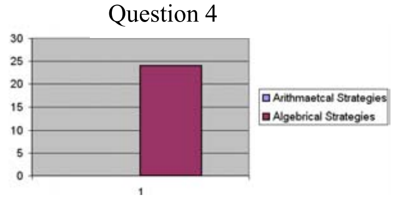
In particular, comparing the two experimental enalisysis, we had the possibility to notice that in Italy *more students than in Slovak, use in this kind of exercise, the arithmetical thought instead of algebraic thought*.

If we consider, for example, the 4thexercise is it possible to see (seeing below the graph-bar) that the arithmetical strategy are present only in the "Gardian" class and only 2 students of this class followed this strategy to solve the question.

In Italy instead, we can find the arithmetical strategy in almost all the classes analized and also in some case the aritmeticl strategies are very relevant (as. IV Ginnasio Liceo Classico Garibaldi).

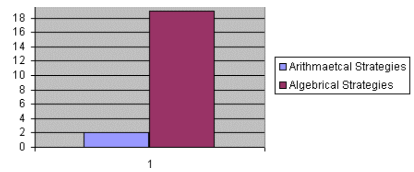
SLOVAK STUDENT 4th exercise

Question 4



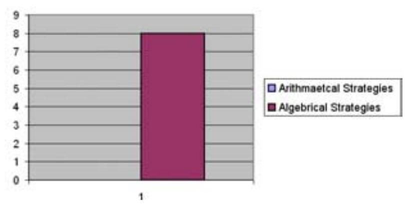
Class: "Zillion"

Question 4



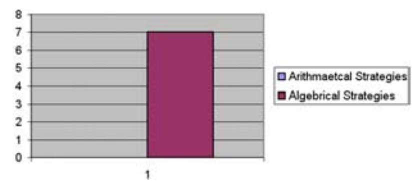
Class: "Gardian"

Question 4



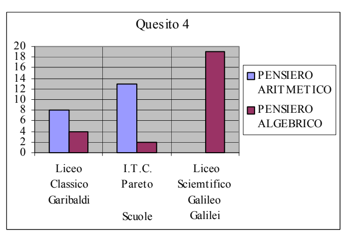
Class: "Riders"

Question 4

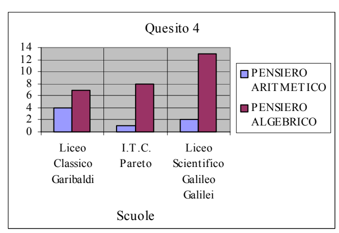


Class: "4.A"

ITALIAN STUDENT[4] 4th exercise

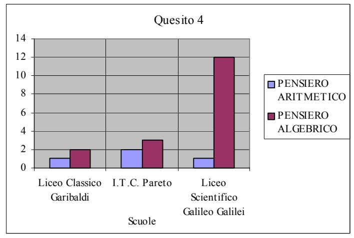


Class: "IV Ginnasio/I Liceo"



Class: "V Ginnasio/II Liceo"

4 *Classes: "IV Ginnasio/I Liceo"*: Liceo Classico Garibaldi (25 students); I.T.C. Pareto (23 students); Liceo Scientifico Galileo Galilei (23 students)
Classes: "V Ginnasio/II Liceo": Liceo Classico Garibaldi (18 students); I.T.C. Pareto (12 students); Liceo Scientifico Galileo Galilei (17 students)
Classes: "I Liceo/III Liceo": Liceo Classico Garibaldi (5 students); I.T.C. Pareto (10 students); Liceo Scientifico Galileo Galilei (15 students)



Class: “I Liceo/III Liceo”

Trying to deep this article and so to underline other relevant results about this first comparison between Italian and Slovak, it would be interesting, according to us, to analyze, in the future, in an other research more detailing, some of the most important results obtained studying the quantitative analysis of the test.

To affect the first type of analysis we could use two different not parametric statistic tools: in particular the implicative analysis of Régis Gras (Chic software 3.1) and the factorial analysis of the correspondences (Software SPSS 11.0).

It could also be interesting to make a qualitative analysis of the protocols of the students putting in evidence some discussion and some explanations of the students during the resolution of same questions presented in the test.

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Appendix A: the Test

Question 1

After reading, solve it and discuss the answer:

A man was 26 years old when his son was born. If you multiply the current ages of the father and his son you get, as result of this operation, the number 456.

What is the current age of the father?

And the age of his son?

Question 2

After reading, solve it and discuss the answer:

A football team bought some balls, 25 Euros everyone.

They received a sale (10 Euros) on the total price.

If this football team paid 240,00 Euros, how many balls bought the team?

Question 3

After reading, solve it and discuss the answer:

Find tree integer and consecutive numbers that the sum of their squares is 50.

Question 4

After reading, solve it and discuss the answer:

Is it true that if you sum to an integer number, the number 4 and after you multiply to number 80 you will have, as result of this operation, 2.360 ?

Question 5

After reading, solve this algebraic equation:

$$2x(x-10)+5(3x^2-4x)=5x(3x-4)-2x(4x-5)$$

Question 6

After reading , solve this algebraic equation:

$$x^2/4 = x/32$$

Question 7

Translate, using variables S (number of students) and P (number of teachers), in a formula, the verbal expression:

<< In this school, students are six time the number of teachers.>>