**Vector calculus and solid geometry at teaching of mathematics at secondary school**

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**Introduction**

One of the long-lasting problems in teaching mathematics at secondary school is the problem of relations among subjects, respectively the continuity of mathematics content and the content of other subjects. However, a specific problem is the continuity or the co-ordination of individual parts of mathematics in the process of education. In this work I want to pay attention to this part of teaching mathematics, specifically geometry - focus on the relation between solid geometry from the vector calculus point of view and the synthetic approach to teaching of solid geometry.

We know that the pupils are taught the basics of vector algebra and analytic geometry already in the 8th or 9th grade at the primary school. Later on they get other information with the notion of vector and its operations (sum, odds, ...) in the first grade at secondary school, and this part is mostly used as a tool for knowledge analytic geometry, for example equations of the straight lines, planes, ... and later, for the analytic geometry of conics, balls, and so on.

Another problem is little attention paid to application of the vector calculus to solve the tasks of solid geometry. These tasks are taught in other parts only by the synthetic geometry, as well as little time spent for application tasks in other subjects (physics, geology, geography, etc.). Students usually don’t find relative with other subjects at solution of solid problems whereupon their knowledge they don’t know or don’t tried to apply.

**Aims of doctoral thesis**

The goal of work is therefore emphasize problems of students at secondary school with solving solid problems, with application their knowledge (from different part of mathematic) in solid geometry, as well as mention problems of students to find connection between parts of mathematics and other learning subjects.

Then the aim of doctoral thesis is also experimental verify validity of the hypotheses H1 – H4 and eventually suggest solutions of effectiveness education of analytical (vectorial) geometry with relating parts of mathematic and thereafter enhancement education between learning subjects at secondary school.
Abovementioned problems we summarized to following hypotheses:

**H1:** The solid geometry is taught at secondary school separately, i. e. students are separately taught axiomatic, separately deal with synthetic geometry, separately analytic geometry, and so the sequence of various approaches is minimal or any.

**H2:** In mathematics textbook there aren’t examples with unification character, which would promote elimination limitations (to fault) listed in H1.

**H3:** Students of secondary school with the established educational schedule haven’t enough ability to apply their knowledge of the vector calculus in other areas of mathematics, apart from analytic geometry, and that in this case only formally.

**H4:** Students of secondary school are not able in ample measure to be aware of the continuity of synthetic and analytic geometry (vector calculus) in solution of particular problem situations. Similar situation applies to University students who will be teachers of mathematics.

**Theoretical framework of problem**

Thesis includes historical and epistemological view of geometry (solid geometry) (the 2nd chapter). We passed through different periods, development of solid geometry (from Euclid, Archimedes till Gaspar Monge) isn’t continual, archaeological excavation mention first contact with arithmetical and geometrical ideas yet stone age. First known notes of solid thought associate with calculating capacity. Her development was episodic and comminuted to smallish discovery motivating needs of arts (Leonardo da Vinci), astronomy (Kepler), architecture (Dürer), optics, surveying, shipbuilding, making of tools and weapons, …

One of the theoretical frameworks represents analysis of educational schedule for teaching geometry (solid geometry) at secondary school. Thesis provides short development and innovation of these educational schedules from about 1868 till actual educational schedule (1997) (the 3rd chapter).

On the 3rd chapter there is too analysis of math textbooks for secondary school, we analysis especially parts of solid geometry. From these analyses we found out that in the math textbooks there aren’t (or there are very minimal) examples with unification character, which would promote elimination limitations (to fault) listed in H1.

In last years in didactic of mathematic have important place exactly theory of didactic situation (Guy Brousseau, Yves Chevallard, Anna Sierpinski, Claire Margolinas, ...), which made other one of the theoretical frameworks (the 3rd chapter). Theory of didactic
situation helps to analysis given didactic problem from different aspects and considers character of problem. Analysis of problem helps us to anticipate if given problem allow pursuing our goals.

**Preparation of the experiment**

In accordance with the tenets theory of didactic situations frame: within the frame of the didactic situation $S_3$ (noosferic didactic situation) we made an analysis of math textbooks for secondary schools, an analysis of various mathematical materials, where the goal was to choose a useful problem for students and which would help us to find out reply to already formulated hypothesis $H1$, $H2$, $H3$ and $H4$ in the introduction. Our goal was to seek such a problem, which the students were not able to solve with the learnt simple algorithms and in math textbook there usually aren’t these types of problems.

At last (after different reflections and after analysis of math textbooks for secondary school) we chose problem from French textbook for secondary school *Mathématiques – Geometrie – Première S-E* as application examples. [61]

The task for the students in this experiment was to try giving an example from the solid geometry with exploitation knowledge out of analytic geometry and vector calculus. (what we awaited results). Results of experiment have mention problems with solving solid problems, with application their knowledge (from different part of mathematic) in solid geometry, as well as mention problems of students to find connection between parts of mathematics and other learning subjects. Results of this experiment had to show how the students could use attainments from these units.

The task was: **Given is a cube ABCDEFGH and K-point, L-point, M-point, N-point, so that K-point is center of upper surface EFGH, L- point is center of the AB, M- point belong to AE, where $|AM| = \frac{1}{3}|AE|$ and N-point belong to BG $|BN| = \frac{1}{3}|BG|$. Are the points K, L, M, and N complanary?**

Based on the above-mentioned criteria the final sentence of the given task can be interpreted in several different ways, their experimental attesting will be the part of our following research of this field.

Studnets could solve problem in different theoretical frame with reference to their level of knowledge and sciental level at secondary school. There are possible strategy
of students solution (analyse a priori of problem designation to experiment) which relate with some parts of mathematic:

Q1: Vector calculus – exploitation collinearity of vectors or complanarity of vectors
Q2: Analytic solution – writing general equation of plane assigned three from four given points and then we prove incidence of fourth point into plane
Q2’: Analytic solution – writing parametric equation of plane assigned three from four given points and then we prove incidence of fourth point into plane
Q2’’: Analytic solution – writing parametric equations of two lines from given fourth points and then we fount out their relative position (if they construct of plane)
Q3: Synthetic approach – construction section of plane of cube and then we prove incidence of fourth point into plane
Q4: Vector calculus – exploitation attributes of “barycentre“.

Possible strategy of students solution - analyse a priori of problem designation to experiment

Q2: Synthetic approach

Pict. 1
1) \( ML \); \( M \in ABF \); \( L \in ABF \); \( ML \in ABF \)

2) \( P, Q; \) \( ML \in ABF \); \( BF \in ABF \); \( EF \in ABF \)
\[ P = ML \cap BF \land Q = ML \cap EF \]

3) \( PN \); \( P \in BCG \); \( N \in BCG \); \( PN \in BCG \)

4) \( X, Y; \) \( PN \in BCG \); \( BC \in BCG \); \( FG \in BCG \)
\[ X = PN \cap BC \land Y = PN \cap FG \]

5) \( LX \); \( L \in ABC \); \( X \in ABC \); \( LX \in ABC \)

6) \( QY \); \( Q \in EFG \); \( Y \in EFG \); \( QY \in EFG \)

7) \( Z; \) \( EH \in EFG \); \( QY \in EFG \)
\[ Z = EH \cap QY \]

8) \( MZ \); \( M \in ADH \); \( Z \in ADH \); \( MZ \in ADH \)

9) \( MLXYZ \)

a) we construct the section plane \( LMN \) of cube \( ABCDEFGH \) and than

b) we attest: \( K \in LMN \) ? (That they are single calculations – for examples: solution with exploitation follows pict. 2 or exploitation resemblance between triangle).
This is parametric equation of plain $\overrightarrow{MLN}(M, \overrightarrow{ML}, \overrightarrow{MN})$: 

\[
x = \frac{1}{3} t \\
y = t + \frac{1}{2} s \\
z = \frac{1}{3} - \frac{1}{3} s ; \ t, s \in R
\]

And then the question is: $K\left[\frac{1}{2}, \frac{1}{2}, 1\right] \in \overrightarrow{MLN}$ ?

We solve the system of equations:

\[
\frac{1}{2} = \frac{1}{3} t \\
\frac{1}{2} = t + \frac{1}{2} s \\
1 = \frac{1}{3} - \frac{1}{3} s
\]

For parameters $t = \frac{3}{2}$ and $s = -2$, the equations have a solution, thereout resulting: 

$K \in \overrightarrow{MLN}$. 
The problem is: Are the vectors $\vec{LS}$ and $\vec{LK}$ collinear? (point S is a centre of line segment $MN$). If the vectors are collinear, so the K-point, L-point, M-point, N-point are complanar.

For the vector $\vec{LS}$ resulting these terms: $\vec{LS} = \vec{LB} + \vec{BN} + \vec{NS} \ ; \ \vec{LS} = \vec{LA} + \vec{AM} + \vec{MS}$

$2\cdot \vec{LS} = \vec{LB} + \vec{LA} + \vec{BN} + \vec{AM} + \vec{NS} + \vec{MS}$.

By means of substitution relations and simple reforms resulting term: $\vec{LS} = \frac{1}{6} (\vec{BG} + \vec{AE})$.

In like the manner for the vector $\vec{LK}$ ⇒ $\vec{LK} = \frac{1}{2} (\vec{BG} + \vec{AE})$.

And from the relationship of vectors $\vec{LS}$, $\vec{LK}$ following: $\vec{LS} = \frac{1}{3} \vec{LK}$ and so, the vectors $\vec{LS}$ and $\vec{LK}$ are collinear.
The base of didactic research is the test students’ reactions in the given didactic situation. So as we could make the test to realize at the first we had to do the full analysis of didactic situation itself.

My working come out from the theory of Guy Brousseau, Yves Chevallard, A. Sierpinska, the foundation of it is the theory of didactic situation. The french school of didactic under leading of G. Brousseaua processed the theory of didactic situation, which is usefull to make analysis of students’ works.

This theory suffers us to analysis the listed problem from the several aspects. And just the analysation of problem in the individual levels of didactic situation form the basis didactic researches.

Part of the experiment is the integration individual phases of the problem solution to system of levels in the analyse didactic situation.
This is the tablet composition of milieu and with it associated types of didactic situations (Margolinias 1994).

<table>
<thead>
<tr>
<th>M₃</th>
<th>Constructional milieu</th>
<th>P₃</th>
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<tbody>
<tr>
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<td>Project milieu</td>
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</tr>
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<td>M₁</td>
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<td>P₁</td>
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<tr>
<td>M₀</td>
<td>Milieu of learning</td>
<td>E₀</td>
<td>Student</td>
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<tr>
<td>M⁻¹</td>
<td>Modeling milieu</td>
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<td>Cognizant intellect st.</td>
<td>P⁻¹</td>
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</tr>
<tr>
<td>M⁻²</td>
<td>Objective milieu</td>
<td>E⁻²</td>
<td>Active student</td>
<td>S⁻²</td>
<td>Modeling situation</td>
</tr>
<tr>
<td>M⁻³</td>
<td>Material milieu</td>
<td>E⁻³</td>
<td>Objective student</td>
<td>S⁻³</td>
<td>Objective situation</td>
</tr>
</tbody>
</table>

We formulated analysis a-priori problem before students’ solutions (our experiment) and analysis includes descending analysis (analysis of teacher’s work), ascending analysis (analysis of student’s work) and we also formulated analysis a-priori problem for statistical program CHIC.

Analyse of teacher’s work (descending analysis)

S₃ – noosferic situation – on this stage we analysis the math textbook for secondary school (2nd or 3rd class), analysis various mathematical materials (educational schedule, …), specifically study of solid geometry, vector algebra and analytic geometry. We want to choose problem, which the students were not able to solve with the learnt simple algorithms and students can use knowledge from different parts of mathematics and other learning subjects. Finish of noosferic situation will be the milieu for the next situation.

S₂ – constructional situation – teacher will try to find examples that were defined in noosferic situation S₃ and on the other side in situation S₁, in which they will be able to realize. Goal was to choose a useful problem for students which would help him to find reply to already formulate hypothesis H₁, H₂, H₃ and H₄. They are examples which students can abet in examples solution Q₁, Q₂, Q₂´, Q₂´´, Q₃ a Q₄.

S₁ – project situation – in situation S₁, teacher writes a text of the example and he “projects” his solution. Student is one on teacher’s consciousness. This is a situation, which involves
student’s activity, too. The student can solve problem in a way that he constructs the section plane of cube, and then he finds out if other point is point of plane; or the student solves a problem that he writes parametric equation of plain and he finds out if the fourth point is the point of the plane; or he applies exploitation collinearity of vectors or complanarity or exploitation “barycentre” (but this solution assuming nothing).

**$S_0$ – didactic situation** – in this situation we analysis and do institucionalition of the new knowledge and we formulate the problem. Teacher takes care of designed goals and he follows student’s solution, too. It is a situation where the analysis of teacher’s work and analysis of student’s work meet, and the didactic situation will be the result of the teaching process.

**Analyse of student’s work (ascending analysis)**

We introduce analysis of problem Q₃ (synthetic approach).

**$S_3$ – objective situation** – the student gets acquainted with the problem and with the material milieu. Material milieu are a cube $ABCDEFGH$ and $K, L, M, N$ – points, cognitive component of milieu are knowledge about incidence of points, lines, planes, geometrical construction new section of body, basis of vertical projection, the notions as skew lines, intersecting lines, etc. Social component of material milieu is minimal because other help isn’t allowing in our experiment (self – activity of students).

**$S_2$ – modelling situation** – the student solves the problem in milieu $S_3$, i. e. he constructs the section plane $\overline{LMN}$ of cube $ABCDEFGH$ (for example). The student makes use of the knowledge from solid geometry, he works with material milieu, he applies visions and plot, he makes use of known relations and practices. Work of student have just character of activity and there teacher don’t interfere with students’ solution. It situation without teacher’s help and feedback and so student must be able to inspect his work and solution.

**$S_1$ – situation of learning** – in this situation the student takes the teacher’s role. He solves the formulated problem by means of question: „Are the given points complanar?“, or he verify to validity $K \in \overline{LMN}$. The student abtains information from reading text of example and he formulats his own results. Student loose character of activity, he is more
thinking and formulating results of his work. The teacher is a scrutator and he tries to help, if the student has some absurdity, alternatively if he fails to solve it. In such a case he falls into position $S_0$ – didactic situation.

$S_0$ – didactic situation – in this situation the work of students is affected by the teacher and takes his advice in form institucionalition, which can help to student by solving given example, but teacher takes to into consideration student’s solution, too. The teacher can help for example with individual elements of section cube or with the correct registration of solving.

**Realization of the experiment. Quantitative analysis of results receives from experiment with exploitation of statistical program CHIC.**

Experiment consisted of one solid problem and was realized in two phases. First participated experiment students from different secondary grammar schools in Nitra (4 classes, in March 2002)). Second phase experiment was realized in one of the secondary grammar schools in Bratislava and also problem-solved students at Faculty of math, physic and informatics in Bratislava (in February 2004). All students had repeat parts of mathematics needs towards solution of our problem. In the aggregate experiment attended 108 students.

On quantitative analysis of results receives from experiment we used statistical program CHIC. In this statistical program were created graphs *Similarity tree, Implicative tree, and Implicative graph.*

CHIC indicated the fact that more frequently students’ solution was synthetic approach and on the other side at least frequently students´ solution was vector solution, which didn’t appear on important level (for results of experiment) in the graphs from CHIC.

Analytic solution used 15 students and 10 of them solved problem with using synthetic approach, too. This fact we can read from Implicative graph although conversion between single variables is only 62 %. From graph the Similarity tree to see altogether equality between thereby that students chose analytic solution of problem and co-ordinate system {alternatively that cock doesn’t fight solve}.

Vector solution appeared only in the graph Similarity tree, but not in important level. But there validated similarity of variables namely student exploit vector calculus and student use notion vector in solution. Only 2 students solved problem with using vector approach and
one of them used synthetic and analytic approach. Hence vector solution didn’t exist in graph Implicative tree and Implicative graph.

No one of 108 student didn’t try solve problem with using facilities of barycentre and so variables of barycentre didn’t appeared in no case of graphs Implicative tree, Implicative graph or Similarity tree. Students usually don’t find relative with other subjects at solution of solid problems whereupon their knowledge they don’t know or don’t tried to apply. Students don’t find relative with physics and mathematics, although notion barycentre students know from lessons of physics from 1st class.

**Conclusion**

Problem at teaching solid geometry is sequence single parts of mathematics within solid geometry. With this problem relate too solutions of solid geometry problems. Concrete, students have problem solve problem, students could solve problem in different theoretical frame with reference to their level of knowledge and sciental level at secondary school, but which the students were not able to solve problem only with the learnt simple algorithms.

The goal of work is mention problems with application students’ knowledge at solving problem from different part of mathematic in solid geometry and also experimental verify validity or invalidity of the hypotheses with using pedagogical experiment.

One of theoretical basis of this work was the educational schedule. Educational schedule goes through some development on the basis of different influences. Our work gives survey of this development and also changes from year 1868 to recent educational schedule issued in 1997. We presented the most important changes, concentrated on educational schedule of teaching geometry (solid geometry) at secondary schools.

By analysis of pedagogical materials, included educational schedule of mathematics for grammar schools (part 3.1) we found out that various parts of geometry in second and third year of study are thought isolated. (There are synthetic geometry, vector calculus and analytical geometry.) This means that sequence of various ways of teaching geometry is minimal. That verified our hypothesis H1.

From analysis of educational schedule mostly from analysis of mathematics’ textbooks for secondary schools (part 3.2) came validity of stated hypothesis H2, that means that in mathematics’ textbooks are very little tasks with unification character. Tasks of this type could help remove problems stated in hypothesis H1, in concrete to find out relations between various themes of solid geometry or to help by applications of knowledge at solving of problem tasks in all thematic parts of solid geometry.
From educational schedule came out that there is not enough time for solving of such, which can help students to be aware of relations of thematic parts or between different teaching subjects. Task given in experiment could have been solved with usage of mass points’ center of gravity, but no one of students tried to solve tasks with usage of knowledge from physics, what we predicted.

Continuance of this work should be from that reason also to use mentioned application’s tasks during teaching process, to acquaint students with these tasks and to solve them in mathematics’ lessons. On approval we put some tasks from solid geometry with their solutions or with instructions how to solve them into tasks’ collection (chapter 6). This collection can be use at teaching of various parts of solid geometry. Tasks can be solved with using of planimetry in space or with finding of some relations between objects in space. We would like also to show, if solving of such tasks would improve teaching of solid geometry, and if students during solving of tasks would find relations between various parts of solid geometry.

In experiment took part 108 students. We predicted, that students would use mostly synthetic and analytic solution of tasks, what showed to be true. Problem didn’t solve 10 students, synthetic solution used 92 students, analytic solution used 15 students and vector solutions used 2 students. In experiment were some students, which wised up to something, that problem having more solutions. So, 10 students solved problem with 2 methods (analytic and synthetic), but only one of them solved problem with using 3 methods (analytic, vector and synthetic). No one of 108 students didn’t try solve problem with using facilities of barycentre.

From these facts follow that students didn’t have ability where to applied knowledge from vector calculus just at analytic geometry, for example equations of the straight lines, planes, ... and later, for the analytic geometry of conics, balls, and so on. And so, in solutions some problem of solid geometry students didn’t apply their knowledge in sufficient measure. Therefore confirmed validity hypothesis H3 and H4.

Experiment was realized also in university (future teachers of mathematics) and confirmed us that there is similar problem with application students’ knowledge from different parts of mathematics in solid geometry and also from other subjects. Whereupon, teacher prefer concrete parts in teaching (in lessons of mathematics) and so teacher often disuse examples with unification character.

Conclusion of the work includes collections of the tasks and we want (of course) expand any interesting application problems (tasks) from solid geometry, which we can
improve on lessons of mathematics. Consequently, correction of collections of these tasks for practical use as part of texts in books of mathematics for secondary schools. In continues of experiment we want to try making educational texts some parts of solid geometry. Educational texts will be especially orientated to coordination between individual parts of solid geometry and rather mention relations among subjects, i. e. mention possibility to applied knowledge from other subjects (especially physics, …) at teaching of solid geometry and at solution of problems from solid geometry.

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Summary

One of the long-lasting problems in teaching mathematics at secondary school is the problem of relations between subjects, respectively the continuity of mathematics content and contents of other subjects. However, a specific problem is the continuity or the co-ordination of individual parts of mathematics in the process of education. I want to pay attention to this part of teaching mathematics, specifically geometry - focus on the relation between solid geometry from the vector calculus point of view and the synthetic approach to teaching of solid geometry.

Another problem is little attention paid to application of the vector calculus to solve the tasks of solid geometry. These tasks are taught in other parts only by the synthetic geometry and there is little time for application of these tasks in other subjects (physics, geology, geography, etc.).

Abovementioned problems we summarized to following hypotheses:

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- to emphasize problems of students at secondary school with solving solid problems, with application their knowledge (from different part of mathematic) in solid geometry, as well as mention problems of students to find connection between parts of mathematics and other learning subjects
- experimental verify validity of the hypotheses
- eventually suggest solutions of effectiveness education of analytical (vectorial) geometry with relating parts of mathematic and thereafter enhancement education between learning subjects at secondary school.