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**Autoreferát dizertačnej práce**

Mental Representations of Pupils about an Open Historical Problem: Goldbach's Conjecture. The Improvement of Mathematical Education from a Historical Viewpoint.
(Mentálne predstavy žiakov v súvislosti s historicky otvoreným problémom: Goldbachova hypotéza. Rozvoj matematického vzdelávania z historického hľadiska)

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Mental Representations of Pupils about an Open Historical Problem: Goldbach's Conjecture. 
The Improvement of Mathematical Education from a Historical Viewpoint.

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Pour un esprit scientifique, toute connaissance est réponse à une question. S'il n'y a pas eu de question, il ne peut y avoir de connaissance scientifique. Rien ne va de soi. Rien n'est donné. Tout est construit.

[For the scientific mind, all knowledge is a response to a question. If there had not been any questions, it would not have been possible to have scientific knowledge. Nothing comes of itself. Nothing is given. Everything is constructed.]


The present situation

Undoubtedly, the intersection of mathematics teaching and the history of mathematics has had a long, fruitful tradition; so, nowadays one of the most frequent questions is: in what way may the history of mathematics influence mathematics education? As E. Barbin has pointed out recently (Barbin, 1996):

Maths teachers who at some point in their career become interested in the history of their subject often report that the understanding they gain influences their teaching or, at the very least, the way in which they perceive mathematics education. These teachers may or may not choose to introduce an historical perspective into their teaching, and they may or may not give their pupils historical texts to read. Nevertheless they say that they have a different view about the errors their pupils make, and they have a better interpretation of certain remarks their pupils make, and are better able to respond to them. They also pay attention to the different stages that have to be passed through in acquiring mathematical knowledge and, in particular, to those obstacles that must be overcome on the way.

If it is true that most knowledge is a response to a question, it is as true that the history of mathematics shows that mathematical concepts are constructed,
modified, and extended in order to solve problems, so an alternative way of writing a history of mathematics is that of a history of problem solving.

The pedagogical value of open problems and conjectures for mathematics teaching is in general remarkable, especially in the educational methodology of problem-solving.

In fact, such a methodology is important above all for the following reasons:
- it allows pupils to use their acquired knowledge to solve problems;
- it improves their logical-deductive abilities;
- it contributes in consolidating knowledge already mastered in a consistent fashion;

Moreover it allows:
- the acquisition of a scientific approach in facing mathematical problems;
- the working out of personal strategies in modeling;
- encourages teamwork;
- forms an antidogmatic mentality by which one can always move ahead.

Thus the aim of this paper is that of analyzing the educational value of mathematical conjectures to improve some pupil's abilities when confronting unsolved questions.

Through facing a conjecture a pupil may be stimulated in acquiring his own ways of reasoning by either following his particular mathematical background or individual intuitive approach in order to solve a question.

We are interested in the following kind of conjecture according to Balacheff (Balacheff, 1994):

*a conjecture is a statement strictly connected with an argumentation and a set of conceptions wherein the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it.*

The relationship between argumentation and proof, strictly connected to the relationship between conjecture and valid statement, has been recently analyzed (Pedemonte, 2000) supposing that, during a solving process, which leads to a theorem, an argumentation activity is developed in order to produce a conjecture.

Instead, in the present case we want to analyze the gradual passage of pupils' attempts from an argumentation to a proof, while they are facing a known unsolved conjecture. We have chosen a historical conjecture like Goldbach's one essentially for the simplicity of its statement and its fascinating empirical evidence.

Goldbach's conjecture states that:

*"Every even number greater than 2 can be represented as the sum of two primes."*

This conjecture belongs to number theory which has a greater number of conjectures than other mathematical fields. On this subject, in the sixth Josiah Willard Gibbs Lecture presented in New York in 1928, the eminent
mathematician G.H. Hardy (1877-1947), who was one of the XXXth century's most famous number-theorist, said (Hardy, 1929):

The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar, the processes of reasoning which it employs are simple, general and few, and it is unique among the other sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same account of „calculus for engineers.“

In the same way, H: Davenport (1907-1969) wrote (Davenport, 1983):

It [number theory] certainly has very few direct applications to other sciences, but it has one feature in common with them, namely the inspiration which it derives from experiment, which takes the form of testing possible general theorems by numerical examples.

So, Goldbach's conjecture seems to be useful in order to point out the following points:
- pupils' conceptions in relation to a conjecture faced during the historical development of mathematics;
- pupils' attempts proving a conjecture reclaimed from history and compared with their argumentative processes;
- to what extent the history of mathematics can favour the study of pupils' conceptions about arguing, conjecturing and proving;
- their reaction to a conjecture's terms seemingly simple to solve;
- their approach in the solving of a conjecture;
- their abilities in carrying out non-standard solving strategies (lateral thinking);

As we know a conjecture can be transformed into a theorem if a proof justifying it is produced; namely, if it is possible to use a mathematical theory allowing the construction of a proof of it.

The basic reason why we have decided to propose an unsolved conjecture like Goldbach's one is, as we have said, to emphasise the role of problems in the historical development of mathematical knowledge. As we know a branch of mathematics maintains mathematicians' interest alive as long as there are always new problems to be solved, because it is only in this way that mathematical knowledge can progress, giving new lymph for the growth of other collateral branches.

It is impossible to do mathematics without asking oneself problems and trying to solve them; or rather the main activity of a mathematician is the solving of problems posed by others or which he puts himself, according to his own tastes and choices. It is in this manner that one can encounter with a really important theorem which enlightens an entire branch of mathematics and through which other trends of search trends are set in motion.
Doubtless, there are really a lot of open problems and unsolved conjectures in number theory, and their number grows yearly, giving continuous inspiration to mathematicians.
b) **Principals targets and the methods selected of the work**

1. **Analysis-a priori of the first experimentation**

The experimental work was carried out by two phases: the first statistical survey was made using a champion of 88 pupils of a high school in Palermo (Sicily) into two final classrooms, precisely into a third and fourth classroom. The students worked in pairs for the part relating to interviews and individually for the production of solution protocols related to the proposed conjecture. The surveyed data were analyzed by the software of inferential statistics CHIC 2000 (Classification Hiérarchique Implicative et Cohésitive) and the factorial statistical survey S.P.S.S. (Statistical Package for Social Sciences). The variables used were 15 and they were the basis for the analysis a-priori of possible answers by pupils.

As an example, here is the analysis a-priori of the problem explained by the following steps:

1) He/she verifies the conjecture by natural number taken at random. (N-random)

2) He/she sums two prime numbers at random and checks if the result is an even number. (Pr-random)

3) He/she factorizes the even number and sums its factors, trying to obtain two primes. (Factor)

4) **Golbach’s method 1**
   He/she considers odd prime numbers less than an even number, summing each of them with successive primes. (Gold1)

5) **Golbach’s method 2 (letter to Euler)**
   He/she writes an even number as a sum of more units, combining these in order to get two primes. (Gold2)

6) **Cantor’s method**
   Given the even number $2n$, by subtracting from it the prime numbers $x \leq 2n$ one by one, by a table of primes one tempts if the obtained difference $2n - x$ is a prime. If it is, then $2n$ is a sum of two primes. (Cant)

7) **The strategy for Cantor’s method**
   He/she considers the primes lower then the given number and calculates the difference between the given number and each of primes. (S-Cant)

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1 The author is grateful to Proff. Marilina Ajello, Carmelo Arena, Egle Cerrone and Emanuele Perez for their helpfulness in carrying out the experimentations into their classrooms.
8) **Euler**
He/she is uneasy to prove the conjecture because one has to consider the additive properties of numbers. (Euler)

9) **Chen Jing-run’s method (1966)**
He/she expresses an even number as a sum of a prime and of a number which is the product of two primes. (Chen)

10) He/she subtracts a prime number from an any even number (lower then the given even number) and he/she ascertains if he/she obtains a prime, so the condition is verified. (Spa-pr)

11) He/she looks for a counter-example which invalidates the statement. (C-exam)

12) He/she considers the final digits of a prime to ascertain the truth of the statement. (Cifre)

13) He/she thinks that a verification of the statement by some numerical examples needs to prove the statement. (V-prova)

14) He/she does not argue anything for the second question. (Nulla)

15) He/she thinks the conjecture is a postulate. (Post)

### 2. Hypothesis of search

The two experimentation were based essentially on the following hypothesis of search, which could be either validated or falsificated:

**I.** Pupils are not able to go beyond the empirical evidence of the conjecture because they do not know how to represent mentally any general method useful for a demonstration.

**II.** Pupils can reach only intuitive conclusions about the validity of Goldbach's conjecture.

### 3. The text for the individual work

The pupils working individually had two hours for answering the following two questions:

Answer the following questions arguing about or motivating every answer:

**a)** Using the enclosed table of primes, the following even numbers can be written as a sum of two primes (in an alone or in a manner more)?

248; 356; 1278; 3896
b) If you have answered the previous question, are you able to prove that it occurs for every even number?

4. The text for the interview for pairs

The interviewees for pairs were made to two pairs of pupils, respectively 16 and 17 aged. In both cases the interview lasted 30 minutes, and it was audio recorded. Here is the text of the interview:

Answer the following question writing only what you have agreed on:

- Is it always true that every even natural number greater than 2 is a sum of two prime numbers?

Let argue about the demonstrative processes motivating them.

The implicative graph of this first experimentation was the following:
5. Pupils' profiles

A further step has been made to sketch a possible profile of a pupil approaching the problem. We have thought that from data three possible profiles of pupils have emerged, and they have been named:

a) **Abdut**: this is the pupil proceeding by *abduction*. Peirce introduced the term *abduction* to indicate the first moment of an inductive process, the one of choosing a hypothesis by which one may explain determined empirical facts.

On the base of such a definition, the pupil named Abdut is who observes how Goldbach's conjecture to be verified in a large number of cases, therefore he supposes it is also valid for any very large even number, and that leads him to the final thesis, that is the conjecture to be valid for every even natural number.

In fact, in this case the pupil proceeds in the following supposed way:

He chooses the strategy of N-random, so he approaches the problem trying to verify it by pairs of natural numbers, chosen at random; he can choose also the strategy of P-random, by choosing a prime number and looking for two natural numbers whose sum is the given prime number. This way of approaching the problem can develop toward the operation of subtracting an even number from a prime in order to obtain an even number, as the Spa-pr strategy, but all these methods can finally persuade him that a proof of the problem is impossible, and so he can fall into the eulerian case.

b) **Intuitionist**: (at the present who proceeds by an inductive argumentation) is instead the pupil having the N-random and Euler strategies in common with Abdut, but thinking that the demonstration of the conjecture can be deduced by a simple numerical evidence, because he is convinced that what happens for the elements of a small finite set of values can be generalized to the infinite set which the small set belongs to; so he uses the V-prova strategy. In short, in an inductive argumentation used by the intuitionist the statement is deduced as a generic case after research from specific cases.

c) **Ipoded**: is just the pupil using a deductive argumentation which can be directly transposed into a deductive demonstration. It is true that he makes some trials and errors, adopting the N-random and P-random strategy as well as Abdut, but soon he follows Chen strategy or he looks for a counter-example whether to demonstrate or to disprove the conjecture.

The implicative graph with the added variables *abdut, intuitionist* and *ipoded* was the following:
6. Some observations about the first experimentation

This first experimentation about Goldbach's conjecture has pointed up that in general most pupils, while facing an unsolved historical conjecture (without knowing it is yet unsolved), start at once with an empirical verification of it which can support their intuition, but after they distinguish theirselves along three different solving tipologies:

- a congruous part of pupils bites off more than one can chew with the following conclusion: since the conjecture is true for all of these particular cases, then it has to be true anyway. These are pupils who have a strong faith in their convictions, but who do not know clearly enough how to pass from an argumentation to a demonstration, by using the achieved data.

- a part of pupils proceeds at the same time by an empirical verification and by an attempt of argumentation and demonstration ending in a mental stalemate. They try to clear a following hurdle: how can I deduce anything general from the empirical evidence? These are pupils who before making any generalization want to be sure of the made steps, therefore they tread carefully.

- few pupils, after a short empirical verification, look at once for a formalization of their argumentations, but if they are not able to do that, they are not diffident about claiming they are in front of something which is undemonstrable. These pupils have a high consideration for their mental processes therefore they think
that if they are not able to demonstrate anything, then it has to be undemonstrable anyway.

By this experimentation we argue that the argumentation favoured by pupils facing a historical conjecture like Goldbach's is the abductive one. Some questions arise from the results which would be advanced by other experimentations:

- Is this result generalizable?
- To what extent it is generalizable?

But the fundamental kernel of this experimentation about the interplay between history of mathematics and mathematics education is that such results could not be pointed out if the analysis a-priori had not been made by the historical-epistemogical remarks which have inspired it.

7. The second experimentation

The second experimentation about Goldbach's Conjecture was made thanks to a group of teachers, coordinated by the author, in some of their classrooms of primary, middle and high school in Piazza Armerina, a provincial town of Enna. The general subject of the experimentation has been about arguing, conjecturing and proving.

a. The experimentation in two primary schools was made in two third classrooms, and it has entailed two different phases by making in a concrete manner:

1th Phase: The following question has been proposed to each pupil by the so-called „Playing evens“ (time: 1 hour):

How can you obtain the first 30 even numbers by putting together prime numbers of the table you have just made?

2th Phase: The experimentation is leaded by little groups of pupils

Can you derive the even numbers obtained by summing always and only two primes? If it is so, can you state this is always the case for an even number?

b. The experimentation's text in the middle school was the following:

The following assertion is always true?

It is always possible to resolve an even number into the sum of two primes?
Let you argue your assertions.

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2 The author is grateful to Prof. Gabriella Termini, Salvatore Marotta, Salvatrice Sorte, Angela Milazzo, Lina Carini, Carmela Buscemi and Fabio Lo Iacona for their helpfulness in carrying out the experimentations into their classrooms.
The analysis a-priori was the following:

1. He/she verifies the conjecture by summing progressive prime numbers and verifying if their sum is an even number or not. [A]

2. He/she chooses an even number and considers prime numbers lesser then it; then he/she verifies the conjecture by choosing one of these prime numbers and noting if its complementary (the difference between the even number and the prime number considered) is also a prime (using tables of primes). [B]

3. He/she resolves the even number into a sum of units; then, he/she applies the associative property until he/she obtains two prime numbers suche that their sum is the given number. [C]

4. He/she resolves the even number into prime factors and sums the factors trying to obtain two primes. [D]

5. He/she verifies the conjecture by considering prime numbers choosen at random. [E]

6. He/she restes on the final figures of a prime number to ascertain the truth of the statement. [F]

7. He/she verifies if the even number is factorable by two primes added to another prime number. [G]

c. The text for the experimentation in two high schools was just the same as the one in the middle school.

The analysis a-priori was the following:

A1: He/she verifies the conjecture by summing progressive prime numbers and verifying if their sum is an even number or not.

A2: He/she chooses an even number and considers prime numbers lesser then it; then he/she verifies the conjecture by choosing one of these prime numbers and noting if its complementary (the difference between the even number and the prime number considered) is also a prime (using tables of primes).

A3: He/she resolves the even number into a sum of units; then, he/she applies the associative property until he/she obtains two prime numbers suche that their sum is the given number.

A4: He/she resolves the even number into prime factors and sums the factors trying to obtain two primes.

A5: He/she verifies the conjecture by considering prime numbers choosen at random.
A6: He/she rests on the final figures of a prime number to ascertain the truth of the statement.
A7: He/she verifies if the even number is factorable by two primes plus another prime number.
A8: He/she verifies the conjecture by taking even numbers at random or progressively.
A9: He/she verifies the conjecture by basing upon the fact that the sum of two odd numbers is always an even number and observing the particularity of number 2, he/she concludes the conjecture is true for even numbers greater than 2.

c) Main conclusions

Writing this work we asked ourselves some questions answering some of them, not others. Our first task was to point up the following points:
- pupils' conceptions in relation to a conjecture faced during the historical development of mathematics;
- pupils' attempts proving a conjecture reclaimed from history and compared with their argumentative processes;
- to what extent the history of mathematics can favour the study of pupils' conceptions about arguing, conjecturing and proving;
- their reaction to a conjecture's terms seemingly simple to solve;
- their approach in the solving of a conjecture;
- their abilities in carrying out non-standard solving strategies (lateral thinking).

Now we are analyzing each of these points trying to deduce some endeavours which will help our educational work.

In order to answer the first two points the first experimentation was very useful, because it pointed out to us a seemingly unexpected conception of pupils about arguing, conjecturing and proving.

In fact, the two initial interviews closed with a strong claim by both of the pairs, namely that Goldbach's conjecture were really a postulate, so an undemonstrable assertion; this is a strong conclusion because it implies that there is a misconception by pupils about the meaning of „postulate“, and it should be advanced by further experimentations.

As for the third point it is clear the role played by history, because without an analysis a-priori based upon historical attempts by mathematicians throughout centuries it should not be possible to analyze profitably pupils' attempts.

Pupils' reaction to terms of Goldbach's conjecture seemingly simple to solve was without any misunderstanding because they knew the meaning of all of the terms involved by the conjecture.
As for pupils' approach in solving of Goldbach's conjecture, both experimentations showed that essentially most of them based on numerical evidence, and only some of them extrapolated their results from a finite set of values to the infinite set of positive integers, but without showing how they passed from trial and error to the conviction of the general validity of Goldbach's conjecture. This is a delicate point which should be advanced by further investigations:
- how do pupils pass from an argumentative phase to the demonstrative one?
- which is the borderline between argumentation and demonstration?
- which is the event that push them from the supposition into the conviction?

The last point, namely lateral thinking, was really what pupils did not use for facing the conjecture, but this is not surprising because they are not generally accustomed to think in a not sequential manner.

Now we turn to the two hypothesis which our work based on proceeding to a verification either of their falsification or not. Recall that they were the following:

I. Pupils are not able to go beyond the empirical evidence of the conjecture because they do not know how to represent mentally any general method useful for a demonstration.

II. Pupils can reach only intuitive conclusions about the validity of Goldbach's conjecture.

Well, throughout the analysis of results one can argue that all of the trials of pupils were implicated by the historical attempt of Golbach. So, this validates the abovewritten two hypothesis.

Moreover, we want to point up that the second experimentation pointed out a characteristic behaviour of pupils from primary to high school while facing Goldbach's conjecture.

First of all it is clear that pupils of primary school cannot proceed if not by a sequential fashion, because they did not yet reach the phase of the demonstration; they were still within a phase of naive argumentation.

Most pupils of middle and high school faced the conjecture by a solving methodology based on random choice and on sequential thinking.

There was a difference between methods of facing the conjecture by pupils of middle and high school. Pupils of middle school in general faced the conjecture basing on an empirical approach, also arguing their choices; but their task went on until a certain point of verification and not beyond.

**On the contrary, there was the presence either of argumentation and attempt of proving in high school pupils' approach. Really many of them tried to infer a prove by their argumentation, and some of them reached also to Chen strategy, wondering all of us. We came down to the same conclusions by the**
results of the two interviews made during the first experimentation. Well, in these cases our initial hypothesis were falsified, and this was a fine surprise.

The results of the experimentation realizes some questions which would be deeped:
- how do pupils get consciousness of a demonstrative process?
- how do pupils get consciousness of the necessity of a demonstrative process?
- how pupils are able to pass from an argumentation to a demonstration?
- are pupils fully conscious of the difference between a verification and a proof?

These and similar questions can give rise to significant experimentations in order to comprise even better metacognitive processes which are basic for the learning phase of pupils and their cultural growth.

But the fundamental kernel of this experimentation about the interplay between history of mathematics and mathematics education is that such results could not be pointed out if the analysis a-priori had not been made by the historical-epistemological remarks which have inspired it.

d) Some of the author's papers are linked to problems of this work


The capital words indicate the contents linking to the thesis

[1] The first paper is devoted to investigate the demonstrative methods of Euclid and Descartes, from a historical viewpoint. It is based on the conviction that going back historically over the stages of the demonstrative thought can be didactally useful in order to understand the methodology of deductive sciences.
[2] The second paper analyzes some educational items into "Il Pitagora", 16
concerning the educational value of **demonstrative and historical methods** for mathematics teaching.


[4] The fourth work is a book on number theory from a historical viewpoint. A whole paragraph is devoted to **Goldbach's conjecture and the attempts of solution**.

e) **Essential Bibliography**

[32] Pedemonte B., *Some cognitive aspects of the relationship between


**Summary**

The aim of this paper is to investigate mental representations of pupils about Goldbach's conjecture for improving the mathematical education from a historical viewpoint. It is based essentially upon two hypothesis of search: the first one concerning their inability to represent mentally any general method useful for a demonstration; the second one concerning their intuitive ability to recognize the validity of the conjecture.

The verification or the invalidation of these hypothesis are very useful in order to understand metacognitive processes which are basic for the learning phase and the cultural growth of pupils.

**Résumé**

Le but de ce papier est étudier les répresentations mentales de les élèves sur la conjecture de Goldbach pou améliorer l'éducation mathématique par un point de vue historique.

Il est basée essentielement sur deux hypothèses de recherche: la premier relatif à la leur incapacité de représenter mentalement une méthode général pour une
démonstration de la conjecture; la second relatif a la leur habilité intuitif sur la validité de la conjecture.
La vérification ou l'invalidation des hypotheses seront utiles pour comprendre les processus metacognitifs qui sont fondamentals pour l'apprentissage et la croissance culturel des élèves.

**Resumé**

Cieľom predloženej práce je štúdium mentálnych procesov žiakov v súvislosti s Goldbachovou hypotézou so zámerom následného skvalitnenia vyučovania matematiky na základe jej historického vývoja. Štúdium je v zásade založené na dvoch hypotézach výskumu: prvá súvisí s neschopnosťou žiakov stanovíť určitú všeobecnú metódu vhodnú pre dôkaz formulovanej hypotézy; druhá v súvislosti s ich intuitívou schopnosťou správne formulovaliť hypotézu.
Proces verifikácie alebo zdôvodnenia nesprávnosti hypotézy budú užitočné k porozumeniu metakognitívnych procesov, ktoré sú základom pre učenie a kultúrny rozvoj žiakov.